Use the Wronskian to prove that the functions $f_1(x) = x^a \cos(b \ln x)$ and $f_2(x) = x^a \sin(b \ln x)$ are linearly independent on $(0, \infty)$ if $a, b \in R$ and $b \neq 0$.

SCORE: _____ / 5 PTS

$$\begin{vmatrix} x^a \cos(b \ln x) & x^a \sin(b \ln x) \\ a x^{a-1} \cos(b \ln x) - b x^{a-1} \sin(b \ln x) & a x^{a-1} \sin(b \ln x) + b x^{a-1} \cos(b \ln x) \end{vmatrix}$$

1 1

$$= \cancel{a x^{2a-1} \cos(b \ln x) \sin(b \ln x)} + b x^{2a-1} \cos^2(b \ln x)$$

$$- \cancel{a x^{2a-1} \cos(b \ln x) \sin(b \ln x)} + b x^{2a-1} \sin^2(b \ln x)$$

$$= b x^{2a-1} \neq 0$$

2 1

Find the general solution of the homogeneous linear differential equation $3x^2 y'' + xy' + y = 0$. **SCORE: _____ / 4 PTS**

$3r^2 + (1-3)r + 1 = 0$

$3r^2 - 2r + 1 = 0$ 1

$r = \frac{2 \pm \sqrt{-8}}{6} = \frac{1}{3} \pm \frac{\sqrt{2}}{3} i$ 1

$y = Ax^{\frac{1}{3}} \cos\left(\frac{\sqrt{2}}{3} \ln x\right) + Bx^{\frac{1}{3}} \sin\left(\frac{\sqrt{2}}{3} \ln x\right)$ 2

Consider the homogeneous linear differential equation $4y'' + 12y' + By = 0$. **SCORE: _____ / 5 PTS**

[a] Find the general solution if $B = 9$.

$4r^2 + 12r + 9 = 0$ $\frac{1}{2}$

$(2r+3)^2 = 0$

$r = -\frac{3}{2}, -\frac{3}{2}$ 1

$y = Ae^{-\frac{3}{2}x} + Bxe^{-\frac{3}{2}x}$ 1

[b] Find the general solution if $B = 25$.

$4r^2 + 12r + 25 = 0$ $\frac{1}{2}$

$(2r+3)^2 + 16 = 0$

$r = \frac{-3 \pm 4i}{2} = -\frac{3}{2} \pm 2i$ 1

$y = Ae^{-\frac{3}{2}x}\cos 2x + Be^{-\frac{3}{2}x}\sin 2x$ 1

Consider the non-homogeneous linear differential equation $3x^2y'' + xy' - 8y = Ax^2$. **SCORE: _____ / 8 PTS**

[a] If $y = x^2 \ln x$ is a particular solution of the equation, find the value of A.

$y' = 2x\ln x + x$ $\frac{1}{2}$

$y'' = 2\ln x + 2 + 1 = 2\ln x + 3$ $\frac{1}{2}$

$3x^2(2\ln x + 3) \quad \cancel{6x^2\ln x} + 9x^2$

$+\, x\,(2x\ln x + x) \quad = +2x^2\ln x + x^2 \quad = 10x^2$ 1 $\quad A = 10$ $\frac{1}{2}$

$-\,8\,(x^2 \ln x) \quad -8x^2\ln x$

-1

[b] **Using superposition and linearity**, find the general solution of $3x^2y'' + xy' - 8y = 5x^2$. $\leftarrow = \frac{1}{2}(10x^2)$

$3r^2 + (1-3)r - 8 = 0$

$3r^2 - 2r - 8 = 0$ $\frac{1}{2}$

$(3r+4)(r-2) = 0$

$r = -\frac{4}{3}, 2$ $\frac{1}{2}$

$y = Ax^{-\frac{4}{3}} + Bx^2 + \frac{1}{2}x^2\ln x$ $\frac{1}{2}$ 1

[c] Solve the initial value problem, $3x^2y'' + xy' - 8y = 5x^2$, $y(1) = 5$, $y'(1) = 3$.

$y' = -\frac{4}{3}Ax^{-\frac{7}{3}} + 2Bx + x\ln x + \frac{1}{2}x$ 1

$5 = A + B$ $\frac{1}{2}$

$3 = -\frac{4}{3}A + 2B + \frac{1}{2}$ $\frac{1}{2}$ MULTIPLY BY 6

MULTIPLY BY 8

$18 = -8A + 12B + 3$

$40 = 8A + 8B$

$\longrightarrow 58 = 20B + 3 \longrightarrow B = \frac{55}{20} = \frac{11}{4}$

$A = 5 - B = \frac{9}{4}$

$y = \frac{9}{4}x^{-\frac{4}{3}} + \frac{11}{4}x^2 + \frac{1}{2}x^2\ln x$

$y_1 = \sin x$ is a solution of $(\tan x)y'' - 3y' + (3\cot x + \tan x)y = 0$. **SCORE: _____ / 8 PTS**

Find a second linearly independent solution.

$y_2 = v\sin x$ 1

$y_2' = v'\sin x + v\cos x$ 1

$y_2'' = v''\sin x + 2v'\cos x - v\sin x$ 1

$y_2 = \sin x\cos x$ 1

$(\tan x)(v''\sin x + 2v'\cos x - v\sin x)$

$-3(v'\sin x + v\cos x)$

$+(3\cot x + \tan x)(v\sin x)$

2

$= v''\sin x\tan x + v'(2\sin x - 3\sin x) + v(-\cancel{\sin x\tan x} - \cancel{3\cos x} + \cancel{3\cos x} + \cancel{\sin x\tan x})$

$= 0 \longrightarrow v''\tan x - v' = 0$ LET $u = v'$

$u'\tan x - u = 0$

$\frac{1}{2}$ $\frac{1}{u}\,du = \cot x\,dx \longrightarrow$ $\frac{1}{2}$ $\ln|u| = \ln|\sin x|$

$\frac{1}{2}$ $u = \sin x = v'$

$\frac{1}{2}$ $v = -\cos x$